An M/G/1 Queue with Second Optional Service and General Randomized Vacation Policy

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Contents

An M/G/1 Queue with Second Optional Service and General Randomized Vacation Policy

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Abstract. This paper studies a continuous time queue system with second optional service where all the arriving customers demand the first "essential" service while only some of them demand the second "optional" service with probability α. The service time of the first essential service and the second optional service both are independent and arbitrarily random variables. Whenever a busy period is completed, the server takes a vacation. If there is at least one customer waiting at a vacation, the server immediately serves the customer. Otherwise, the server takes another vacation with probability p, or remains idle with probability $1-p$. We give some performances analysis of this model. Finally, it gives some numerical examples to illustrate the effect of the probabilities λ and p on the mean system size, waiting time, the probabilities when the server is idle and is on a vacation.

Keywords: Continuous time queue · Second optional service · General randomized vacation policy · Supplementary variable method

1 Introduction

As soon as the first essential service of a customer is completed, he or she immediately leaves the system with probability α or accepts the second optional service with probability $1-\alpha$. This service policy is called second optional service policy and was firstly studied by Madan [1]. The literature discussed an M/G/1 queue with the second optional service in which the first essential service time follows a general distribution, but the second optional service is assumed to be exponentially distributed. Medhi [2] extended Madan's model by considering that the second optional service follows a general distribution. Wang [3] examined an M/G/1 queue with second optional service and breakdowns in which the first

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essential service time follows a general distribution, but the second optional service is assumed to be an exponential distribution. In addition, there are many other queue models concerned second optional service which have been studied in recent years, details of which may be seen [4–18].

When a busy period is completed, the server immediately takes a vacation. The server will serve the customers if there are customers waiting in the queue at the end of a vacation. Otherwise, the server either remains idle with probability p or takes another vacation with probability $1-p$. This pattern continues until the server has taken J vacations. The server keeps idle, if there are not customers in the system at J^{th} vacation. This vacation policy is called randomize vacation policy and was studied by Ke [10]. However, some more complex queue systems with this policy are hard to analysis, as in a queue system with working vacations. Therefore, we cancel the limit of randomized vacation policy, namely the server remains idle with probability p or takes another vacation with probability $1-p$ if no customers are waiting for service at the end of any vacation, and then let the pattern continue forever. Here we define this vacation policy as general randomized vacation policy. The policy eliminates a parameter J so that it is easy to be widely applied to some more complex queue systems. Moreover it is not a stand alone vacation policy but also summarizes multiple and single vacation policy. That is our motivation to put forward the general randomized vacation policy.

The remainder of this paper is organized as follows. A full description of the model and analysis of the system embedded with the Markov chain are given in Sect. 2. In Sect. 3, some important measures performance of the system are obtained. In Sect. 4, we give two special cases of the model. Finally in Sect. 5, we present some numerical results to illustrate the effect of α and p on the performance of the system. Section 6 concludes the paper.

2 Description and Analysis of Model

In the section, we describe our model with following assumptions. Customers arrive the system according to a Poisson process with rate λ. When the first "essential" service of a customer is completed by the server, he or she will demand the second "optional" service with probability α. We assume that the first "essential" service and the second "optional" service both follow general distributions, with probability distribution functions $G_1(x)$ and $G_2(x)$, respectively. In addition, let $g_k(x)$, $\frac{1}{u_k}$, and $u_k(x)dx = \frac{dG_k(x)}{1-G_k(x)}, k = 1, 2$, denote the corresponding probability density functions, means and hazard rate functions. When an busy period is completed, the server immediately takes a vacation with general distribution $V(x)$. Let $v(x), v$ and $w(x)dx = \frac{dV(x)}{1-V(x)}$ be the corresponding probability density function, mean and hazard rate function. If there is at least one customer in the system at the end of the vacation, the server will immediately serve the customer. Otherwise, the server will either take another vacation with probability p or remain idle waiting for the arrival of customers with probability $1-p$. Obviously, if $p = 1$, our model can be simplified to the

M/G/1 queue with second optional service and multiple vacations; if $p = 0$, the model can be also simplified to the M/G/1 queue with second optional service and single vacation.

We assume, throughout this paper, that various stochastic processes involved in the system are mutual independence and obey first-come first-served (FCFS) service discipline. For a given function $F(x)$, its Laplace-Stieltjes transform (LST) denotes by $F^*(s) = \int_0^\infty e^{-sx} dF(x)$. And then, we define $\rho = \frac{\lambda}{u_1} + \alpha\frac{\lambda}{u_2}$. Obviously, $\rho < 1$ is the necessary and sufficient condition when a steady state solution exists.

Let $N(t)$ be the system size including the one being served (if any) at time t, and denote by $G_1^-(x), G_2^-(x)$ and $V^-(x)$ the elapsed first "essential" service, elapsed second "optional" service and elapsed vacation at time t, respectively. In addition, we introduce the following random variable

$$J(t) = \begin{cases} 0, & \text{if the server is idle at time t,} \\ 1, & \text{if the server is busy providing a essential service at time t,} \\ 2, & \text{if the server is busy providing a second optional service at time t,} \\ 3, & \text{if the server is taking a vacation at time t} \end{cases}$$

At time t, the system can be described by the process $(N(t), c(t))$ where $c(t) = 0$ if $J(t) = 0$; $c(t) = G_1^-(x)$ if $J(t) = 1$; $c(t) = G_2^-(x)$ if $J(t) = 2$ and $c(t) = V^-(x)$ if $J(t) = 3$. For further studying the model, we define the following limiting probabilities:

$$\begin{aligned} p_{0,0} &= \lim_{t\to\infty} p(N(t) = 0, c(t) = 0), \\ p_{1,n} &= \lim_{t\to\infty} p(N(t) = n, c(t) = G_1^-(x); x \le G_1^-(x) \le x + dx),\ n \ge 1,\ x \ge 0, \\ p_{2,n} &= \lim_{t\to\infty} p(N(t) = n, c(t) = G_2^-(x); x \le G_2^-(x) \le x + dx),\ n \ge 1,\ x \ge 0, \\ p_{3,n} &= \lim_{t\to\infty} p(N(t) = n, c(t) = V^-(x); x \le V^-(x) \le x + dx),\ n \ge 0,\ x \ge 0 \end{aligned}$$

Then in steady-state condition, the Kolmogorov forward equations to govern the model can be written as follows:

$$\lambda p_{0,0} = (1 - p) \int_0^\infty p_{3,0}(x) w(x) dx \tag{1}$$

$$\frac{dp_{1,1}(x)}{dx} + [\lambda + u_1(x)] p_{1,1}(x) = 0 \tag{2}$$

$$\frac{dp_{1,n}(x)}{dx} + [\lambda + u_1(x)] p_{1,n}(x) = \lambda p_{1,n-1}(x),\ n \ge 2 \tag{3}$$

$$\frac{dp_{2,1}(x)}{dx} + [\lambda + u_2(x)] p_{2,1}(x) = 0 \tag{4}$$

$$\frac{dp_{2,n}(x)}{dx} + [\lambda + u_2(x)] p_{2,n}(x) = \lambda p_{2,n-1}(x),\ n \ge 2 \tag{5}$$

$$\frac{dp_{3,0}(x)}{dx} + [\lambda + w(x)] p_{3,0}(x) = 0 \tag{6}$$

$$\frac{dp_{3,n}(x)}{dx} + [\lambda + w(x)]p_{3,n}(x) = \lambda p_{3,n-1}(x),\ n \geq 1 \tag{7}$$

Equations (1)–(7) will be solved under the following boundary conditions at time $x = 0$

$$p_{1,1}(0) = \lambda p_{0,0} + (1-\alpha)\int_0^\infty p_{1,2}u_1(x)dx + \int_0^\infty p_{2,2}u_2(x)dx + \int_0^\infty p_{3,1}w(x)dx \tag{8}$$

$$p_{1,n}(0) = (1-\alpha)\int_0^\infty p_{1,n+1}u_1(x)dx + \int_0^\infty p_{2,n+1}u_2(x)dx + \int_0^\infty p_{3,n}w(x)dx,\ n \geq 2 \tag{9}$$

$$p_{2,n}(0) = \alpha\int_0^\infty p_{1,n}u_1(x)dx,\ n \geq 1 \tag{10}$$

$$p_{3,0}(0) = (1-\alpha)\int_0^\infty p_{1,1}u_1(x)dx + \int_0^\infty p_{2,1}u_2(x)dx + p\int_0^\infty p_{3,0}w(x)dx \tag{11}$$

In order to solve the above Equations, we define some probability generating functions as follows:

$$P_i(x,z) = \sum_{n=1}^{\infty} p_{i,n}(x)z^n,\ P_3(x,z) = \sum_{n=0}^{\infty} p_{3,n}(x)z^n, P_i(z) = \int_0^\infty P_k(x,z)dx$$

where $i = 1, 2; k = 1, 2, 3$.

Multiplying both sides of Eqs. (2) and (3) by z^n $(n = 1, 2, \cdots)$ and summing over n, then we have

$$P_1(x,z) = P_1(0,z)[1 - G_1(x)]e^{-\lambda(1-z)x} \tag{12}$$

Similar proceeding on the Eqs. (4)–(7), then we obtain

$$P_2(x,z) = P_2(0,z)[1 - G_2(x)]e^{-\lambda(1-z)x} \tag{13}$$

and

$$P_3(x,z) = P_3(0,z)[1 - V(x)]e^{-\lambda(1-z)x} \tag{14}$$

In the same way, we can get the following equation from Eqs. (8) and (9)

$$\begin{aligned} P_1(0,z) = \lambda p_{0,0}(z-1) - p_{3,0}(0) + \frac{1-\alpha}{z}P_1(0,z)G_1^*(\lambda(1-z)) \\ +\tfrac{1}{z}P_2(0,z)G_2^*(\lambda(1-z)) + P_3(0,z)V^*(\lambda(1-z)) \end{aligned} \tag{15}$$

For convenience, let $r(z) = \lambda(1-z)$. From Eq.(15), we have

$$\begin{aligned} P_1(0,z) = \lambda p_{0,0}(z-1) - p_{3,0}(0) + \frac{1-\alpha}{z}P_1(0,z)G_1^*(r(z)) \\ +\tfrac{1}{z}P_2(0,z)G_2^*(r(z)) + P_3(0,z)V^*(r(z)) \end{aligned}$$

Solving the differential Eq.(6) yields

$$p_{3,0}(x) = p_{3,0}(0)(1 - V(x))e^{-\lambda x} \tag{16}$$

Then multiplying both sides of Eq.(16) by $w(x)$ and integrating with x from 0 to ∞, together with Eq.(1), we have

$$p_{3,0} = \frac{\lambda p_{0,0}}{(1-p)V^*(\lambda)} \tag{17}$$

Substituting Eq.(17) into Eq.(15), we obtain

$$\begin{aligned} P_1(0,z) = \lambda p_{0,0}(z-1) - \tfrac{\lambda p_{0,0}}{(1-p)V^*(\lambda)} + \frac{1-\alpha}{z} P_1(0,z) G_1^*(r(z)) \\ + \frac{1}{z} P_2(0,z) G_2^*(r(z)) + P_3(0,z) V^*(r(z)) \end{aligned} \tag{18}$$

Since $P_3(0,z) = p_{3,0}(0)$, Eq.(18) can be written as follows:

$$\begin{aligned} P_1(0,z) = \lambda p_{0,0}(z-1) - \tfrac{\lambda p_{0,0}}{(1-p)V^*(\lambda)} + \frac{1-\alpha}{z} P_1(0,z) G_1^*(r(z)) \\ + \frac{1}{z} P_2(0,z) G_2^*(r(z)) + \frac{\lambda p_{0,0}}{(1-p)V^*(\lambda)} V^*(r(z)) \end{aligned} \tag{19}$$

Multiplying both sides of Eq. (10) by z^n $(n = 1, 2, \cdots)$ and summing over n, then we have

$$P_2(0,z) = \alpha P_1(0,z) G_1(r(z)). \tag{20}$$

Substituting Eq.(20) into Eq.(19), we obtain

$$P_1(0,z) = \frac{\lambda z p_{0,0}[1 + (1-p)V^*(\lambda)(1-z) - V^*(r(z))]}{(1-p)V^*(\lambda)[(1-\alpha)G_1^*(r(z)) + \alpha G_1^*(r(z))G_2^*(r(z)) - z]} \tag{21}$$

Integrating both sides of Eq.(12) with x from 0 to ∞, then we get

$$P_1(z) = P_1(0,z) \frac{1 - G_1^*(r(z))}{\lambda(1-z)} \tag{22}$$

Substituting Eq.(21) into Eq.(22), we have

$$P_1(z) = \frac{z p_{0,0}[1 - G_1^*(r(z))][1 + (1-p)V^*(\lambda)(1-z) - V^*(r(z))]}{(1-p)V^*(\lambda)(1-z)[(1-\alpha)G_1^*(r(z)) + \alpha G_1^*(r(z))G_2^*(r(z)) - z]} \tag{23}$$

Performing similar operations on Eqs. (13) and (14), then we get

$$P_2(z) = P_2(0,z) \frac{1 - G_2^*(r(z))}{\lambda(1-z)} \tag{24}$$

and

$$P_3(z) = P_3(0,z) \frac{1 - V^*(\lambda(1-z))}{\lambda(1-z)} \tag{25}$$

Then, substituting Eqs. (20) and (17) into (24) and (25), respectively, we have

$$P_2(z) = \frac{p_{0,0} z \alpha G_1^*(r(z))[1 - G_2^*(r(z))][1 + (1-p)V^*(\lambda)(1-z) - V^*(r(z))]}{(1-p)V^*(\lambda)(1-z)[(1-\alpha)G_1^*(r(z)) + \alpha G_1^*(r(z))G_2^*(r(z)) - z]} \tag{26}$$

and

$$P_3(z) = \frac{p_{0,0}[1 - V^*(r(z))]}{(1-p)V^*(\lambda)(1-z)} \tag{27}$$

From Eqs. (23), (26) and (27), we get the probability generating function for steady-state system size

$$\begin{aligned} P(z) &= P_1(z) + P_2(z) + P_3(z) + p_{0,0} \\ &= \frac{p_{0,0}[1 + (1-p)V^*(\lambda)(1-z) - V^*(r(z))]G_1^*(r(z))[(1-\alpha) + \alpha G_2^*(r(z))]}{(1-p)V^*(\lambda)[(1-\alpha)G_1^*(r(z)) + \alpha G_1^*(r(z))G_2^*(r(z)) - z]} \end{aligned} \tag{28}$$

Using the normalization condition $P_1(z) + P_2(z) + P_3(z) + p_{0,0} = 1$, thus we have

$$p_{0,0} = \frac{(1-p)(1-\rho)V^*(\lambda)}{\lambda v + (1-p)V^*(\lambda)} \tag{29}$$

Substituting $p_{0,0}$ into Eq. (28), it is given as

$$P(z) = \frac{(1-\rho)[1 + (1-p)V^*(\lambda)(1-z) - V^*(r(z))]G_1^*(r(z))[(1-\alpha) + \alpha G_2^*(r(z))]}{[\lambda v + (1-p)V^*(\lambda)][(1-\alpha)G_1^*(r(z)) + \alpha G_1^*(r(z))G_2^*(r(z)) - z]}$$

Based on the above analysis, we will give some performance analysis for the system in the next section.

3 Performance Analysis

In the section, we will obtain the probability generating function of the steady state system size at a departure epoch, and the mean values for the steady state system size, waiting time, sojourn time. In addition, we will obtain the probability for each state of the server.

We denote by $\pi_n, n = 0, 1, \cdots$ the probabilities that there are n customers in the system at a departure point (no including the one just departing from the system). Then, we can obtain the forward equations as follows:

$$\pi_n = M(1-\alpha)\int_0^\infty p_{1,n+1}u_1(x)dx + M\int_0^\infty p_{2,n+1}u_2(x)dx, \ n = 0, 1, \cdots \tag{29}$$

where M is the normalizing constant.

Multiplying Eq.(29) by z^n $(n = 1, 2, \cdots)$ and summing over n, then together with Eqs. (12) and (13), we obtain the probability generating function of the system size $\Pi(z)$ at a departure epoch as follows:

$$\Pi(z) = \frac{M\lambda p_{0,0}[1 + (1-p)V^*(\lambda)(1-z) - V^*(r(z))]G_1^*(r(z))[(1-\alpha) + \alpha G_2^*(r(z))]}{(1-p)V^*(\lambda)[(1-\alpha)G_1^*(r(z)) + \alpha G_1^*(r(z))G_2^*(r(z)) - z]} \tag{30}$$

by utilizing the normalizing condition $\Pi(1) = 1$, from Eq. (30), we have

$$M = \frac{(1-\rho)(1-p)V^*(\lambda)}{\lambda p_{0,0}[\lambda v + (1-p)V^*(\lambda)]} \tag{31}$$

Substituting Eq. (31) into Eq. (30), we obtain

$$\Pi(z) = \frac{(1-\rho)[1+(1-p)V^*(\lambda)(1-z) - V^*(r(z))]G_1^*(r(z))[(1-\alpha)+\alpha G_2^*(r(z))]}{[\lambda v + (1-p)V^*(\lambda)][(1-\alpha)G_1^*(r(z)) + \alpha G_1^*(r(z))G_2^*(r(z)) - z]} \tag{32}$$

Thus, the probability generating function of the steady state system size at a departure epoch is same as the one of the system size at a random epoch. From the Eq. (32), we can have a theorem as follows:

Theorem 1. *If $\rho < 1$, the steady-state system size L can be decomposed into the sums of two stochastic variables, i.e., $L = L_0 + L_d$, where L_0 denotes the steady-state system size at departure epoch of M/G/1 queue with second optional service whose generating function has been given in [1], L_d is the steady-state additional system size due to the general randomized vacations with the probability generating function as follows*

$$L_d(z) = \frac{1+(1-p)V^*(\lambda)(1-z) - V^*(r(z))}{(1-z)[\lambda v + (1-p)V^*(\lambda)]} \tag{33}$$

Proof. From Eq. (32), it is very easy to obtain the theorem.

Utilizing Theorem 1, we can obtain a remark as follows.

Remark 1. If $\rho < 1$, the mean system size can be written as $E[L] = E[L_0] + E[L_d]$, where $E[L_0]$ denotes the mean system size at departure epoch of M/G/1 queue with second optional service whose detailed expression has been given in [1], $E[L_d]$ is the additional mean system size due to the general randomized vacations with the detailed expression as follows

$$E[L_d] = \frac{\lambda^2 v^{(2)}}{2[\lambda v + (1-p)V^*(\lambda)]} \tag{34}$$

where $v^{(2)}$ stands for the two moment of the general distribution $V(x)$.

Utilizing Remark 1 and Little formula, we can obtain the other two remarks as follows.

Remark 2. If $\rho < 1$, the expected value for the sojourn time of a customer in the system is given by

$$E[W] = \frac{E[L_0]}{\lambda} + \frac{\lambda v^{(2)}}{2[\lambda v + (1-p)V^*(\lambda)]} \tag{35}$$

Remark 3. If $\rho < 1$, the expected value for the waiting time of a customer in the system is given by

$$E[W_q] = \frac{E[L_0]}{\lambda} + \frac{\lambda v^{(2)}}{2[\lambda v + (1-p)V^*(\lambda)]} - \frac{1}{u_1} - \frac{\alpha}{u_2} \tag{36}$$

From the expressions of $P_1(z), P_2(z), P_3(z)$ and $p_{0,0}$, we can determined the probability for each state of the server, as in the following Corollary 1.

Corollary 1. *If $\rho < 1$, then*

(1) the probability when the server is idle is

$$p_{0,0} = \frac{(1-p)(1-\rho)V^*(\lambda)}{\lambda v + (1-p)V^*(\lambda)}$$

(2) the probability when the server is busy with supplying the first essential service is

$$P_1 = \rho_1$$

(3) the probability when the server is busy with supplying the second optional service is

$$P_2 = \rho_2$$

(4) the probability when the server is taking a vacation is

$$P_3 = \frac{(1-\rho)\lambda v}{\lambda v + (1-p)V^*(\lambda)}$$

where $\rho_1 = \dfrac{\lambda}{u_1}$, $\rho_2 = \dfrac{\alpha\lambda}{u_2}$.

4 Special Cases of the Model

In the section, we will give two special cases of our model by choosing the different value of p. We will only study $\Pi(z)$ for the two cases of the model, and the other parameters can be studied similarly.

Case 1. Let $p = 1$. Then our model can be simplified to the M/G/1 queue with second optional service and multiple vacations. Let $p = 1$ in $\Pi(z)$. We have the probability generating function of system size at a departure epoch as follows

$$\Pi(z) = \frac{(1-\rho)[1 - V^*(r(z))]G_1^*(r(z))[(1-\alpha) + \alpha G_2^*(r(z))]}{\lambda v[(1-\alpha)G_1^*(r(z)) + \alpha G_1^*(r(z))G_2^*(r(z)) - z]}$$

Case 2. Let $p = 0$. Then our model can be simplified to the M/G/1 queue with second optional service and single vacation. In addition, let $p = 0$ in $\Pi(z)$. We have the probability generating function of system size at a departure epoch as follows

$$\Pi(z) = \frac{(1-\rho)[1 + V^*(\lambda)(1-z) - V^*(r(z))]G_1^*(r(z))[(1-\alpha) + \alpha G_2^*(r(z))]}{[\lambda v + V^*(\lambda)][(1-\alpha)G_1^*(r(z)) + \alpha G_1^*(r(z))G_2^*(r(z)) - z]}$$

5 Numerical Results

In the section, our first purpose is to study the effects of parameters p and λ on the expected system size of messages and the expected waiting time of messages in the system. We assume that the length of a first essential service, a second optional service and a vacation all follow exponential distributions with parameters μ_1, μ_2 and ν, respectively.

For convenience, we choose $\mu_1 = 2.5, \mu_2 = 2.0, \nu = 1.5, \alpha = 0.5$ and $p = 0, 0.2, 0.5, 0.7, 1$, and then vary the value of λ from 0 to 1.0.

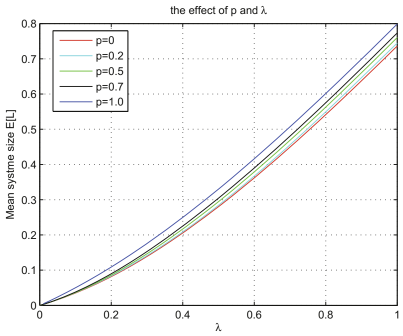

Fig. 1. The expected system size

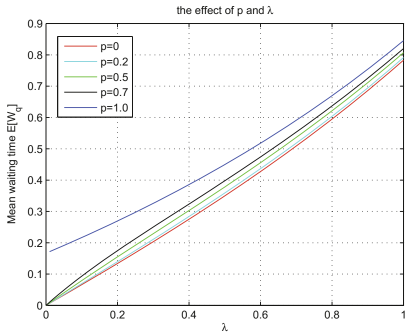

Fig. 2. The expected waiting time

Figures 1 and 2 show that the expected system size and the expected waiting time are functions of the arrival rate λ and p. We find that whenever λ increases, the expected system size and expected waiting time increase at a higher level with a fixed p, so the both are increasing functions of λ. Similarly the both are also increasing functions of p with a fixed λ.

The second purpose is to study the effects of parameters p and λ on probabilities $p_{0,0}$ and P_3. We make some assumptions as above.

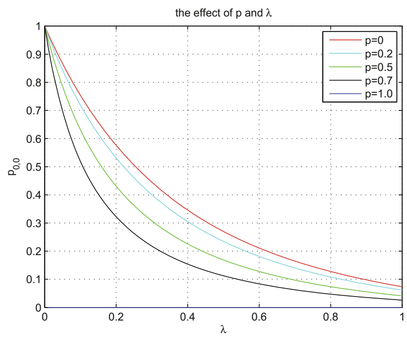

Fig. 3. The expected system size

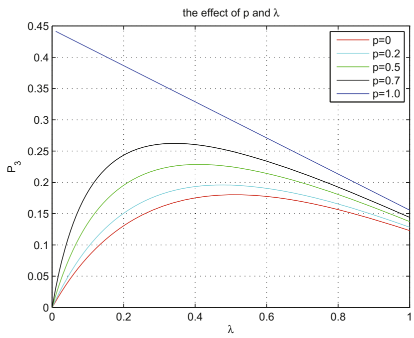

Fig. 4. The expected waiting time

Figures 3 and 4 show that $p_{0,0}$ is a function of the arrival rate λ and p. We find that λ increases, $p_{0,0}$ decreases at a lower level with a fixed p, so it is a decreasing function of λ. Furthermore, P_3 is increasing function about p with a fixed λ, but not of the monotonicity, of λ with a fixed p.

6 Conclusions

In this paper, we study the general randomized vacation policy for the M/G/1 queueing system with second optional service. By the Kolmogorov forward equations and supplementary variable method, we obtain the probability generating functions for the steady state system size and expected values for the steady state system size, waiting time and sojourn time. Additionally, utilizing numerical illustration, we study the effects of parameters p and λ on the expected system size of messages, the expected waiting time of messages and the probabilities when the server is idle and is on vacation.

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